SETS OF RANDOM EVENTS WITH GIVEN I/D STRUCTURE

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How can we construct a probability space and define a set of n random events with prescribed i/d-structure ?

Definition 1 (i/d-structure)**.** Take $\{A_1, ..., A_n\}$ a set of n random events. Make the configuration of all combinations of events from level $n = 2$ to n. $k = 1$ is trivial. $2 \leq k \leq n$ gives $(n-1)$ level, where each level k contains all $\binom{n}{k}$ combinations of size k. For any, check the product property, write 1 if satisfied, 0 if not. Count the number of satisfied product property at each level k. Denote by i_k the number of true product relations at level k. Count the number of unsatisfied product property at each level k. Denote by $d_k = \binom{n}{k} - ik$ the number of true product relations at level k. $(i_2, i_3, ..., i_n)$ is the i/d-structure.

Illustration

$k = n$	□□□□□□□□□□□□□	$0 \leq i_n \leq \binom{n}{n} = 1$
$k = n-1$	□□□□□□ ... ... □□□□□□	$0 \leq i_{n-1} \leq \binom{n}{n-1}$
...	...	...
$k = 3$	□□□ □□□ ... ... □□□ □□□	$0 \leq i_3 \leq \binom{n}{3}$
$k = 2$	□□ □□ ... ... □□ □□	$0 \leq i_2 \leq \binom{n}{2}$
$k = 1$	□ □ □ □ ... ... □ □ □ □	$0 \leq i_1 \leq \binom{n}{1}$

There are two boundary cases :

1. If all the $2^n - n - 1$ product properties are satisfied, it means that we have mutually independence of random events.

Illustration

$k = n$	□□□□□□□□□□□□□	1
...	...	...
$k = 3$	□□□ □□□ ... ... □□□ □□□	1 1 ... ... 1 1
$k = 2$	□□ □□ ... ... □□ □□	1 1 ... ... 1 1
$k = 1$	□ □ □ □ ... ... □ □ □ □	1 1 1 1 ... ... 1 1 1 1

2. If all the $2^n - n - 1$ product properties are not satisfied, it means that we have totally dependence of random events.

Illustration

$k = n$	□□□□□□□□□□□□□	0
...	...	...
$k = 3$	□□□ □□□ ... ... □□□ □□□	0 0 ... ... 0 0
$k = 2$	□□ □□ ... ... □□ □□	0 0 ... ... 0 0
$k = 1$	□ □ □ □ ... ... □ □ □ □	0 0 0 0 ... ... 0 0 0 0

Does it exist a probability space and a set $\{A_1, ..., A_n\}$ of n random events in this space such that the i/d-structure of $\{A_1, ..., A_n\}$ is exactly $(i_2, ..., i_n)$? Yes!

Main problem: Stoyanov (2013)

Suppose that we are given a natural number $n \geq 2$ and $(n-1)$-tuple $(i_2, ..., i_n)$ where $0 \leq i_k \leq \binom{n}{k}$. Does it exist a probability space and a set $\mathcal{X}$ of n binary random variables in this space such that the i/d-structure of $\mathcal{X}$ is exactly $(i_2, ..., i_n)$?

Solution

Let $\mathcal{I} = \{1, 2, ..., n\}$ and $\mathcal{M}_{\max} = M : M \subset \mathcal{I}, 2 \leq |M| \leq n$, $|\mathcal{M}_{\max}| = 2^n - n - 1$. The following stronger theorem stipulates

Theorem 1. Let $\mathcal{M}$ be an arbitrary subset of $\mathcal{M}_{\max}$ with $|\mathcal{M}| \geq 2$, then there is a probability space $(\Omega, \mathcal{F}, P)$ and a set of n random events, say $\mathcal{A}_n = \{A_1, ..., A_n\}$ in this space such that, for $k = 2, ..., n$ and $1 \leq j_1 < ... < j_k \leq n$, the product relation $P(A_{j_1} ... A_{j_k}) = P(A_{j_1}) ... P(A_{j_k})$ is satisfied if and only if $(j_1, ..., j_k) \in \mathcal{M}$.

Example 1. Take $n = 3$, $\mathcal{A} = \{A_1, A_2, A_3\}$, a prescribed i/d-structure (i_2, i_3), $0 \leq i_2 \leq \binom{3}{2}$, $0 \leq i_3 \leq \binom{3}{3}$, $|\Omega| = 2^3 = 8$, $\mathcal{I} = \{1, 2, 3\}$.

We construct smaller events ω_*, let $\Omega = \{\omega_\varnothing, \omega_1, \omega_2, \omega_3, \omega_{12}, \omega_{13}, \omega_{23}, \omega_{123}\}$. Denote $P(\omega_*) = \Pi_*$, let $P = \{\Pi_\varnothing, \Pi_1, \Pi_2, \Pi_3, \Pi_{12}, \Pi_{13}, \Pi_{23}, \Pi_{123}\}$, let $A_1 = \{\omega_1, \omega_{12}, \omega_{13}, \omega_{123}\}$, $A_2 = \{\omega_1, \omega_{12}, \omega_{23}, \omega_{123}\}$ and $A_3 = \{\omega_3, \omega_{13}, \omega_{23}, \omega_{123}\}$ implies

$$\begin{aligned} P(A_1) &= \Pi_1 + \Pi_{12} + \Pi_{13} + \Pi_{123}, \\ P(A_2) &= \Pi_2 + \Pi_{12} + \Pi_{23} + \Pi_{123}, \\ P(A_3) &= \Pi_3 + \Pi_{13} + \Pi_{23} + \Pi_{123} \end{aligned}$$

Assume that $P(A_i) = \frac{1}{2}$, the i/d-structure $i_2 = 1$, $i_3 = 1$ and more precisely $\mathcal{M} = \{13, 123\}$. Let $\mathcal{M}' = \mathcal{M} \cup \{\varnothing, 1, 2, 3\}$ six elements in $\mathcal{M}'$ giving six equations (8 unknowns)

$$\begin{aligned} \Pi_\varnothing + \Pi_1 + \Pi_2 + \Pi_3 + \Pi_{12} + \Pi_{13} + \Pi_{23} + \Pi_{123} &= 2^0 = 1 \\ \Pi_1 + \Pi_{12} + \Pi_{13} + \Pi_{123} &= 2^{-1} \\ \Pi_2 + \Pi_{12} + \Pi_{23} + \Pi_{123} &= 2^{-1} \\ \Pi_3 + \Pi_{13} + \Pi_{23} + \Pi_{123} &= 2^{-1} \\ \Pi_{13} + \Pi_{123} &= 2^{-2} \\ \Pi_{123} &= 2^{-3} \end{aligned}$$

with solutions

$\Pi_\varnothing = \alpha + \beta,\ \Pi_1 = \frac{1}{4} - \alpha,\ \Pi_2 = \frac{3}{8} - \alpha - \beta,\ \Pi_3 = \frac{1}{4} - \beta,\ \Pi_{12} = \alpha,\ \Pi_{13} = \frac{1}{8},\ \Pi_{23} = \beta,\ \Pi_{123} = \frac{1}{8}$

If $\alpha = \beta = \frac{1}{8}$, $\mathcal{A}$ become mutually independent, $\Pi_H = \frac{1}{8}$ for every $H \subset \mathcal{I}$ and the inequations $P(A_1A_2) \neq 2^{-2}$, $P(A_2A_3) \neq 2^{-2}$ implies $\alpha \neq \frac{1}{8}$, $\beta \neq \frac{1}{8}$, giving $\mathcal{H} = \{(\alpha, \beta) : 0 < \alpha < \frac{1}{4},\ 0 < \beta < \frac{1}{4},\ 0 < \alpha + \beta < \frac{3}{8}\}$

General case

$\Omega = \{\omega_H : H \subset \mathcal{I}, |\Omega| = 2^n,\ P = \{\Pi_H : \Pi_H = P(\omega_H),\ H \subset \mathcal{I}\}\}$

$A_j = \{\omega_H : i \in H,\ H \subset \mathcal{I}\}$, assume $P(A_j) = \frac{1}{2}$ for each j prescribed $\mathcal{M} \subseteq \mathcal{M}_{\max}$.

Let $\mathcal{M}' = \mathcal{M} \cup \{\varnothing, 1, 2, .., n\}$. For each

$$M \in \mathcal{M}',\ P(\cap_{j \in \mathcal{M}} A_j) = \sum_{\mathcal{M} \subset H} \Pi_H = 2^{-|M|} \tag{1}$$

gives 2^n unknowns, say $|\mathcal{M}'|$ equations.

Let the solution form a hyperplane ξ in the space R^{2^n}, and it is easy to find that the point

$\tilde{\Pi} = \{\Pi_H = \frac{1}{2^n},\ H \in \mathbb{Z}^{\mathcal{I}}\} \in \xi$.
1. When $\mathcal{M} = \mathcal{M}_{\max}$, $\tilde{\Pi}$ is the solution we find and in this case $\mathcal{A}$ is mutually independent.
2. $\mathcal{M} \subsetneqq \mathcal{M}_{\max}$, $|\mathcal{M}'| < 2^n$, $\tilde{\Pi}$ is not the solution we find but we may have many solutions in its ε-neighborhood in the hyperplane ξ.

Proof. **a.** $\dim\ \xi = 2^n - |\mathcal{M}'|$. Note equation (1) implies $Q\Pi = \beta$. In this formula, Q is the coefficient matrix $(Q_{M,H})$, size $|\mathcal{M}'|{\times}2^n$ where $Q_{MH} = \begin{cases} 1 & \mathcal{M} \subset H \\ 0 & \text{otherwise} \end{cases}$, $\Pi = \begin{pmatrix} \Pi_\varnothing \\ \Pi_1 \\ . \\ . \\ . \end{pmatrix}$, $\beta = \begin{pmatrix} 2^0 \\ 2^{-1} \\ . \\ . \\ . \end{pmatrix}$.
According to the knowledge in advanced algebra we know $\dim\ \xi = 2^n - \text{rank(Q)}$. Now prove that $\text{rank(Q)} = |\mathcal{M}'|$.
If there is a vanishing linear combination of the rows defined with some weights, say γ_M for $M \in \mathcal{M}'$, then for every $H \subset \mathcal{I}$, we have

$$\sum_{M \in \mathcal{M}'} \gamma_M \cdot Q_{MH} = \sum_{M \in H} \gamma_M = 0$$

There exists a minimal set $M^* \in \mathcal{M}'$ and for all $M \subset M^*$, $M \in \mathcal{M}'$ implies $M = M^*$. Let $H = M^*$, $\gamma_{M^*} = 0$. Deleting M^* from $\mathcal{M}'$ and repeating leads to $\gamma_M = 0$ for all $M \in \mathcal{M}'$. So $\dim\ \xi = 2^n - \text{rank(Q)} = 2^n - |\mathcal{M}^*|$.
b. The dimension of the sub-hyperplane ξ' we need to leave out of ξ is $_textdim - 1$, $\sum_{M' \subset H} \Pi_H \neq 2^{-|M'|}$ for all $' \notin \mathcal{M}'$ there are $2^n - |\mathcal{M}'|$ inequations. Let $s = 2^n - |\mathcal{M}'|$, $\mathcal{M}_{\max} \setminus \mathcal{M}' = \{M_1, M_2, ..., M_s\}$.

$$\begin{cases} (1) \\ \sum_{\mathcal{M}_i \subset H} \Pi_H = 2^{-|\mathcal{M}_i|} \end{cases} \tag{2}$$

Its solution space is noted as ξ_i.
Obviously, there are $|\mathcal{M}'|{+}1$ equations in (2), so according to **a.**, $\dim(\xi_i) = 2^n - |\mathcal{M}'| - 1 = \dim\ \xi - 1$. We make $\xi' = \cap_{i=1}^s \xi_i$ for each $\dim(\xi_i) < \dim\ \xi$. When we leave ξ' out of ξ there are still lots of points left and they are actually the solutions we want. □

Example 2. Bernstein example
Suppose that we have four tickets : $\boxed{112}$ $\boxed{121}$ $\boxed{211}$ $\boxed{222}$
We choose randomly one ticket and consider the events
$A_1 = \{\text{1 occurs in the 1st place}\}$
$A_2 = \{\text{1 occurs in the 2nd place}\}$
$A_3 = \{\text{1 occurs in the 3rd place}\}$
$P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$

Illustration
$k = 3$ $\boxed{A_1 | A_2 | A_3}$ $P(A_1A_2A_3) = 0 \neq \frac{1}{8} = P(A_1)P(A_2)P(A_3)$
$k = 2$ $\boxed{A_1 | A_2}$ $\boxed{A_1 | A_3}$ $\boxed{A_2 | A_3}$ $P(A_1A_2) = P(A_1A_3) = P(A_2A_3) = \frac{1}{4}$
At level $k = 2$, all the 3 product properties are satisfied, $i_2 = 3$, $d_2 = 0$. At level $k = 3$, there are no product properties satisfied, $i_3 = 0$, $d_3 = 1$. The i/d-structure of the collection $\{A_1, A_2, A_3\}$ is $(3, 0)$ and we also get that pair-wise independence does not imply mutual independence.

Example 3. Take 4 random events $\{A_1, A_2, A_3, A_4\}$. We suppose that :

- the product property is satisfied at level $k = 4$, which means that

$$P(A_1A_2A_3A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$$

- the product property is satisfied for any $\binom{4}{2}$ pairs of events
- the product property is not satisfied for any of the $\binom{4}{3}$ triplets of events

We have this configuration

$k = 4$ $\quad \boxed{A_1}\boxed{A_2}\boxed{A_3}\boxed{A_4} \quad i_4 = 1$

$k = 3$ $\quad \boxed{A_1}\boxed{A_2}\boxed{A_3} \quad \boxed{A_1}\boxed{A_2}\boxed{A_4} \quad \boxed{A_2}\boxed{A_3}\boxed{A_4} \quad \boxed{A_1}\boxed{A_3}\boxed{A_4} \quad i_3 = 0$

$k = 2$ $\quad \boxed{A_1}\boxed{A_2} \quad \boxed{A_1}\boxed{A_3} \quad \boxed{A_2}\boxed{A_3} \quad \boxed{A_2}\boxed{A_4} \quad \boxed{A_3}\boxed{A_4} \quad i_2 = 6$

Such a set of events is dependent and the i/d-structure of the collection $\{A_1, A_2, A_3, A_4\}$ is given as $(6, 0, 1)$.

References

Stoyanov, J. M. (2013). Counterexamples in probability, https://store.doverpublications.com/products/9780486499987?srsltid=AfmBOopJtXwcQhJchiShHL8GVoKkyMvvvRZkaX8see7zKCAauEXENGIO.